The gluing construction for normally generic J-holomorphic curves

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Abstract. Let $C_0 = [\Sigma_0, f_0]$ be a stable J-holomorphic curve in an almost complex manifold (V, J). Under an assumption of normal genericity, we show that C_0 has in the space of homologous stable J-curves of the same genus a locally Euclidean neighbourhood of the expected dimension given by Riemann-Roch. In dimension 4, the normal genericity condition is satisfied if i) $\langle c_1(TV), C_i\rangle > 0$ for each component of C_0 and ii) the only singularities of C_0 are ordinary double points. We also give an extension of these results for curves containing a given finite set. As an application, we show that every symplectic surface of degree 3 in $\mathbf{CP}^2$ is symplectically isotopic to an algebraic curve.

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1. Introduction and statement of the results

The gluing technique for J-holomorphic curves has been introduced by A. Floer in his definition of "symplectic Floer homology" [F]. It first appeared in the following form: given two (or more) J-holomorphic curves which intersect and satisfy a suitable genericity condition, their union can be approximated by a continuous family of J-holomorphic curves, the convergence near the intersection points being like that of $\{xy = t\} \subset \mathbf{C}^2$ to $\{xy = 0\}$ for $t \in \mathbf{C}^*$, $t \to 0$. The proof is similar to C. Taubes' gluing construction for instantons, ie one first constructs an approximately J-holomorphic curve and then one uses the implicit function theorem to find a true J-holomorphic curve nearby. This is very well explained in the Appendix A to the book [MS] of D. McDuff and D. Salamon.

[MS] treated only the case where all curves are rational and the intersections take place between different curves, in which case the deformed curve is also rational: cf. the model $\{xy = 0\} \to \{xy = t\}$, now considered in $\mathbf{CP}^2$. Later it was realized that this gluing construction can be generalized to any J-curve parameterized by a Riemann surface with nodes (or *nodal*): in other words, the components may have arbitrary genus, and have self-intersection. The idea of parameterizing a J-curve by a nodal Riemann surface is due to M. Kontsevich [KM], who used it to reformulate Gromov's compactness theorem in terms of *stable* J-holomorphic-maps, viewing it as an "embedded" version of Deligne-Mumford's compactification of $\mathcal{M}_g$.

Remark. We prefer to use the terminology "stable J-holomorphic *curve*", or simply "stable J-curve".

To formulate this generalized gluing result, we introduce the space $\overline{\mathcal{M}}_g(V, J, A)$ of stable J-curves of genus g in V, in the homology class $A \in H_2(V; \mathbf{Z})$. A point in this space is an isomorphism class $[\Sigma_0, f_0]$, where Σ_0 is a nodal Riemann surface of genus g and $f_0 : \Sigma_0 \to V$ is a J-holomorphic map, with $(f_0)_*(\Sigma_0) = A$. The stability condition means that the group $\Gamma_0 = \mathrm{Aut}(\Sigma_0, f_0)$ is finite. If $\Gamma_0 = \{\mathrm{Id}\}$, f_0 is called simple (or primitive). We shall denote by $\mathcal{M}_g(V, J, A)$ the subspace obtained when Σ_0 is smooth.

We shall also need the extended spaces

$$\overline{\mathrm{M}}(V) = \bigcup_{J\in\mathcal{J}(V)} \overline{\mathcal{M}}_g(V, J)\ ,\ \overline{\mathrm{M}}_g(V, A) = \bigcup_{J\in\mathcal{J}(V)} \overline{\mathcal{M}}_g(V, J, A),$$

where $\mathcal{J}(V)$ is the space of almost complex structures on V. They are equipped with natural continuous projections $\pi : \overline{\mathrm{M}}_g(V)\,,\,\overline{\mathrm{M}}_g(V,A) \to \mathcal{J}(V)$.

Remark. Ideally, one would like to consider structures of class $\mathcal{C}^\infty$, but for the Fredholm analysis and in particular the use of the Sard-Smale theorem, one has to work with structures of class $\mathcal{C}^{N,\alpha}$ with N very large (cf. [MS])).

Let $C_0 = [\Sigma_0, f_0]$ be an element of $\overline{\mathcal{M}}_g(V,J,A)$. Linearizing the J-holomorphy equation $\overline{\partial}_J f = 0$ for maps $f : \Sigma_0 \to V$ defined on the fixed Riemann surface Σ_0, one obtains the operator

$$D_{f_0} : \Gamma(f_0^* TV) \to \Omega^{0,1}(f_0^* TV),$$

which is of the form $\overline{\partial}+a$ (generalized Cauchy-Riemann). It is Fredholm (for suitable differetiability classes), with complex index (half the real index) given by Riemann-Roch:

$$\mathrm{ind}_{\mathbf{C}}(D_{f_0}) = \langle c_1(TV), A\rangle + n(1-g).$$

Here $c_1(TV) \in H^2(V;\mathbf{Z})$ is the first Chern class of (TV,J). Note that the formal dimension $i(A,g)$ of $\overline{\mathcal{M}}_g(V,J,A)$ (or of $\mathcal{M}_g(V,J,A)$) over $\mathbf{C}$ is

$$i(A,g) = \langle c_1(TV), A\rangle + (n-3)(1-g) = \mathrm{ind}(D_{f_0}) + (3g-3),$$

the difference $3g-3$ being $\dim(\mathcal{M}_g) - \dim\,\mathrm{Aut}(\Sigma_0)$, accounting for the variation of the source as a Riemann surface and the equivalence relation.

We shall say that $C_0 = [\Sigma_0, f_0]$ is *parametrically generic* if D_{f_0} is onto. We can now state the generalized gluing result. With different notations, it can be found in the papers [RT] of Y. Ruan and G. Tian (Theorem 6.1, special case $\nu_t = 0$), [LT] of J. Li and G. Tian (Proposition 3.4, special case $S = 0$) and [FO] of K. Fukaya and K. Ono ([FO], Theorem 12.9, special case $E_\sigma = 0$). It is also implicit in the work [Sie1] of B. Siebert.

Theorem 0. *Let $C_0 = [\Sigma_0, f_0]$ in $\overline{\mathcal{M}}_g(V,J,A)$ be parametrically generic. Then there is a local homeomorphism*

$$\phi_J : (\overline{\mathcal{M}}_g(V,J,A), C_0) \to (\mathbf{C}^m \times (\mathbf{C}^{i(A,g)-m}/\Gamma_0)\,,\,(0,[0])),$$

where the finite group $\Gamma_0 = \mathrm{Aut}(\Sigma_0, f_0)$ acts linearly. Moreover, let us number the nodes $1,\cdots,m$, and let $\tau_1,\cdots,\tau_m$ be the m first complex coordinates of ϕ_J. Then the curves C such that $\tau_i(C) = 0$ are exactly those for which the parameterizing surface Σ keeps the i-eth node of Σ_0.

Finally, there exists a local homeomorphism

$$\widetilde{\phi} : (\overline{M}_g(V,A), C_0) \to (\mathbf{C}^m \times (\mathbf{C}^{i(A,g-m)}/\Gamma_0) \times \mathcal{J}(V)\,,\,(0,[0],J))$$

whose restrictions $\phi_{J'}$ all have the same properties as ϕ_J.

Actually, the papers [LT], [FO], [Sie1] also treat the case when D_{f_0} is not onto (in particular for non-simple curves, where this condition is no longer generic in J), by introducing a complement to the image and studying an equation of the type $\overline{\partial}_J f = g$ where g belongs to this complement: the solutions have then nothing to do with J-holomorphic curves, though of course counting them may be interesting from the point of view of symplectic geometry (general definition of Gromov-Witten

invariants or of Floer homology). Here we have a different point of view since we are interested in J-holomorphic curves *per se*.

There is another possible linearization of the equation $\overline{\partial}_J f = 0$, in which one also allows the complex structure of Σ_0 to vary (cf. [IS]). Considering it as an almost complex structure j on Σ_0, one differentiates the equation $\overline{\partial}_J f = \frac{1}{2}(df + J\, df\, j)$ with respect to j as well as to f. At the point (j_0, f_0), we get an operator

$$\begin{aligned} \widetilde{D}_{f_0} &: \Omega^{0,1}_{j_0}(T\Sigma_0) \times \Gamma(f_0^* TV) \to \Omega^{0,1}(f_0^* TV) \\ \widetilde{D}_{f_0}(v, \xi) &= \frac{1}{2} J\, df_0(v) + D_{f_0}(\xi). \end{aligned}$$

Let us call $C_0 = [\Sigma_0, f_0]$ *normally generic* if $\widetilde{D}_{f_0}$ is onto. The terminology is due to the fact that, when it is possible to define a normal linearized $\overline{\partial}$-operator D^{N_0} as in ([G], 2.1.C$_1$, [HLS], [IS]), the surjectivity of $\widetilde{D}_{f_0}$ is equivalent to that of D^N: see section 4.

The main result of this paper is that the gluing construction works for a normally generic curve.

Theorem 1. *$C_0 = [\Sigma_0, f_0]$ be a stable J-curve which is normally generic. Then all the conclusions of Theorem 0 hold.*

In dimension 4, the normal genericity can be ensured by homotopical assumptions ([G], 2.1.C$_1$, [HLS], [IS]). We need here a slight generalization for a curve parametrized by a nodal surface.

Proposition 1. *Let (V, J) be an almost complex manifold of dimension 4, and let $[\Sigma_0, f_0] = C_0$ be a stable J-curve in $\overline{\mathcal{M}}_g(V, J)$. We assume that f_0 is an embedding with distinct tangents near the nodes, and that the restriction f_i to each component Σ_i satisfies*

$$\langle f_i^* c_1(TV), \Sigma_i \rangle > |df_i^{-1}(0)| \ , \ i = 1, \cdots, r.$$

Then C_0 is normally generic.

The application of Theorem 1 to this case is most significant when f_0 is an embedding with distinct tangents at the nodes, since then one obtains a description of all geometrically close J-curves. A J-curve for which f_0 is an embedding with distinct tangents at the nodes will be called *nodal*. Combining Proposition 1 with Theorem 1 and the adjunction formula, we obtain

Corollary 1. *Let C_0 in $\overline{\mathcal{M}}_g(V, J, A)$ be a nodal J-curve in dimension 4, with components $C_1, \cdots, C_r$ in the image, satisfying*

$$\langle c_1(TV), C_i \rangle > 0 \ , \ i = 1, \cdots, r.$$

Then there are local homeomorphisms

$$\begin{aligned} \phi_J &: (\overline{\mathcal{M}}_g(V, J, A), C_0) \to (\mathbf{C}^m \times \mathbf{C}^{d(A)-m} \, , (0, [0])), \\ \widetilde{\phi} &: (\overline{M}_g(V, A), C_0) \to (\mathbf{C}^m \times \mathbf{C}^{d(A)-m} \times \mathcal{J}(V) \, , (0, [0], J)) \end{aligned}$$

where

$$d(A) = \frac{1}{2}(A.A + \langle c_1(TV), A \rangle),$$

with properties as above.

These results can be extended to the case when the curve is required to contain a given finite set $F \subset V$ in the image. We denote $\overline{\mathcal{M}}_g(V, J, A)$ the subset of such curves, and by $\widetilde{D}_{f_0,F}$ the restriction of $\widetilde{D}_{f_0}$ to the subspace

$$\{(v, \xi) \in \Omega^{0,1}(T\Sigma_0) \times \Gamma(f_0^* TV) \mid \xi = 0 \text{ on } f_0^{-1}(F)\}.$$

Then Theorem 1 generalizes to Theorem 1', and Proposition 1 generalizes to Proposition 1' (Section 6). Corollary 1 then becomes

Corollary 2. *Let C_0 in $\overline{\mathcal{M}}_g(V, J, A)$ be a nodal J-curve in dimension 4, with components $C_1, \cdots, C_r$ in the image. Let F be a finite subset of the smooth part of the image. Assume that*

$$|F \cap C_i| < \langle c_1(TV), C_i \rangle \ , \ i = 1, \cdots, r.$$

Then there are local homeomorphisms

$$\begin{aligned} \phi_J^F : (\overline{\mathcal{M}}_g(V, J, A; F), C_0) &\to (\mathbf{C}^m \times \mathbf{C}^{d(A)-|F \cap C_0|} \, , \, (0, [0])) \\ \widetilde{\phi}_F : (\overline{M}_g(V, A; F), C_0) &\to (\mathbf{C}^m \times \mathbf{C}^{d(A)-|F|} \times \mathcal{J}(V) \, , \, (0, [0], J)) \end{aligned}$$

with properties as above.

As an application of this last result, we give a partial result on the problem of isotopy classes of symplectic surfaces in $\mathbf{CP}^2$.

Theorem 3. *A symplectic surface of degree 3 in $\mathbf{CP}^2$ is symplectically isotopic to an algebraic curve.*

Organization of the paper. In section 2, we recall the definition of stable J-holomorphic curves and give a local description of these objects.

In section 3 we prove Theorem 1, following the method of [MS]. For the continuity of the constructions with respect to $\vec{t} \in \mathbf{C}^m$, we follow [RT].

In section 4 we define (under some mild restrictions, and only in dimension 4) the normal $\overline{\partial}$-operator and prove that its surjectivity is equivalent to the normal genericity.

In section 5 we recall some important properties in dimension 4 and use them to prove Proposition 1 and thus Corollary 1.

In section 6 we state and prove the results about curves containing a fixed finite subset: Theorem 1', Proposition 1' and Corollary 2.

Finally, in section 7 we study the isotopy problem for symplectic surfaces and and prove Theorem 3.

2. The space of stable J-curves of genus g in V

Let (V, J) be an almost complex manifold.

2.1. Definition. The space $\overline{\mathcal{M}}_g(V, J)$ of stable J-holomorphic curves of genus g in V is the set of isomorphism classes $[\Sigma, f]$, where

(i) Σ is a connected Riemann surface with nodes, of arithmetic genus $h^1(\Sigma, \mathcal{O}_\Sigma) = g$; recall that this is also the genus of any connected smooth deformation, and is given by

$$g = \sum_{i=1}^{r} (\widetilde{g}_i - 1) + m + 1 = \sum_{i=1}^{r} \widetilde{g}_i + m - r + 1,$$

where r is the number of components, $\widetilde{g}_1, \cdots, \widetilde{g}_r$ the genera of the normalizations and m the number of nodes

(ii) $f : \Sigma \to V$ is a J-holomorphic map

(iii) $\mathrm{Aut}\,(\Sigma, f)$ is finite (stability condition); equivalently, f is nonconstant on each unstable component in the sense of Deligne-Mumford (ie rational and with at most 2 points of the normalization lying over the nodes, or of genus 1 with no point over the nodes).

If A is an element of $H_2(V; \mathbf{Z})$, then $\overline{\mathcal{M}}_g(V, J, A)$ is the subset of $\overline{\mathcal{M}}_g(V, J)$ defined by the condition $f_*([\Sigma]) = A$.

We give now a local description of the topology of $\overline{\mathcal{M}}_g(V, J)$ near a point $C_0 = [\Sigma_0, f_0]$, following [FO] (sections 9 and 10) and [Sie1] (sections 2 and 3).

We define the topology by describing a countable fundamental system of neighbourhoods of $[\Sigma_0, f_0]$. For this, we choose an arbitrary Riemannian metric μ on V, so that the area

$$\mathrm{area}_\mu(f) = \int_\Sigma |\Lambda^2 df|$$

of any map $f : \Sigma \to V$ is defined. We also choose an arbitrary metric on Σ_0 and a sequence $\epsilon_i > 0$, $\epsilon_i \to 0$.

Then by definition, the neighbourhood $\mathcal{N}_i$ $(i \in \mathbf{N})$ of $[\Sigma_0, f_0]$ is the set of all curves $[\Sigma, f]$ such that there exists a continuous map $\kappa : \Sigma \to \Sigma_0$ with the following properties:

i) ϕ is a diffeomorphism over $\Sigma_0 \backslash N_0$, where N_0 is the set of nodes, and $\phi^{-1}(N_0)$ is a union of circles

ii) $|\overline{\partial}\phi| \leq \epsilon_i |\partial\phi_i|$ over $\Sigma_0 \backslash \mathcal{U}_{\epsilon_i}$ where $\mathcal{U}_{\epsilon_i}$ is a ϵ_i-neighbourhood of N_0

iii) $\mathrm{dist}_{C^0}(f \circ \phi^{-1}, f_0) \to 0$ on $\Sigma_0 \setminus N_0$.

Remarks

1) Note that if V is a point then $\overline{\mathcal{M}}_g(V, J) = \overline{\mathcal{M}}_g$, the Deligne-Mumford compactification.

2) We have given a rather weak description of the convergence of f_i to f. Actually, the proof of Gromov's compactness theorem shows that one will also have, for some $\epsilon'_i \to 0$, $\mathrm{dist}_{C^1}(f \circ \phi^{-1}, f_0) < \epsilon'_i$ on $\Sigma_0 \setminus \mathcal{U}_{\epsilon'_i}$ and $\mathrm{area}_\mu(f_{|E_i}) < \epsilon'_i$ for each component E of $\mathcal{U}_{\epsilon'_i}$. In fact, we shall need the fact that, in a suitable sense, f is close to f_0 in the L^p_1-topology for any $p \in]2, +\infty[$.

3) One can generalize the definition of $\overline{\mathcal{M}}_g(V, J)$ to include marked points, thus obtaining spaces $\overline{\mathcal{M}}_{g,m}(V, J)$. As in 1), one has $\overline{\mathcal{M}}_{g,m}(V, J) = \overline{\mathcal{M}}_{g,m}$ if V is a point.

Clearly, the topology defined in this way does not depend on the choice of (ϵ_i) or μ. Less obvious, but true, is the fact that it is Hausdorff. Another easy property is that the homology class $A = f_*([\Sigma]) \in H_2(V; \mathbf{Z})$ is locally constant on $\overline{\mathcal{M}}_g(V, J)$, so that one has a topological sum

$$\overline{\mathcal{M}}_g(V, J) = \coprod_{A \in H_2(V; \mathbf{Z})} \overline{\mathcal{M}}_g(V, J, A).$$

To state Gromov's compactness theorem, we recall the notion of tameness: an almost complex manifold (V, J) equipped with a metric μ is *tame* if for every compact $K \subset V$ and every $C > 0$, there is a compact $K' \subset V$ such that every J-curve $C = f(\Sigma)$, Σ connected, which meets K, is contained in K'. This is the case for instance if V is compact, or (V, J, μ) has a compact quotient, or more generally has a bounded geometry in a suitable sense.

Theorem. *Let (V, J, μ) be a tame almost complex manifold. Then the area functional* area_μ *defines a proper map on $\overline{\mathcal{M}}_g(V, J)$.*

In practice, an important special case is the following.

Corollary. *Let (V, ω, J) be a compact symplectic manifold equipped with an almost complex structure which is ω-positive. Then all the spaces $\overline{\mathcal{M}}_g(V, J, A)$, $A \in H_2(V; \mathbf{Z})$, are compact.*

2.2. Explicit description of J-curves close to $[\Sigma_0, f_0]$

Following [FO], [LT] and [Sie], we explicitly describe the curves $[\Sigma, f]$ which are close to some fixed point $[\Sigma_0, f_0]$ in $\overline{\mathcal{M}}_g(V, J)$.

1) One defines, for $\vec{t} \in \mathbf{C}^n$ small enough, the differentiable surface

$$\Sigma_{\vec{t}} = ((\Sigma_0 \backslash \mathcal{U}_{\vec{t}}) \cup \mathcal{A}_{\vec{t}})_{/\sim} ,$$

where

$$\mathcal{U}_{\vec{t}} = \bigcup_{i=1}^{m} \phi_i(\Delta_{|t_i|} \times \{0\} \cup \{0\} \times \Delta_{|t_i|}),$$

the ϕ_i being holomorphic embeddings from $\Delta \times \{0\} \cup \{0\} \times \Delta$ to Σ_0 whose images are disjoint neighbourhoods of the nodes, and

$$\mathcal{A}_{\vec{t}} = \coprod_{i=1}^{m} \psi_i(A_{t_i}) \ , \ A_t = \{(x, y) \in \Delta^2 \mid xy = t\}.$$

We identify $\psi_i(x, y)$ with $\phi_i(x, 0)$ for $|x| > |y|$, and with $\phi_i(0, y)$ for $|x| < |y|$.

2) For every $v \in H^1(T\Sigma_0)$ small enough, one defines on $\Sigma_{\vec{t}}$ the almost complex structure (necessarily integrable) $j_{\vec{t},v}$ as follows. First, for $\vec{t} = \vec{0}$ one chooses a lift σ of the natural map $\Omega^{0,1}(T\Sigma_0) \to H^1(T\Sigma_0)$, which is equivariant under Γ_0 and such that each $\sigma(v)$ vanishes on $\mathcal{U}_1$. Then, denoting j_0 the almost complex structure on Σ_0, one sets

$$j_v = (\text{Id} - \frac{j_0\sigma(v)}{2})\, j_0\, (\text{Id} - \frac{j_0\sigma(v)}{2})^{-1}$$

so that the cohomology class of $\frac{\partial j_v}{\partial v}_{|v=0} = \widetilde{v}$ is v (Kodaira-Spencer map). Then there is a unique $j_{\vec{t},v}$ such that $j_{\vec{t},v} = j_v$ on $\Sigma_0 \backslash \mathcal{U}_{\vec{t}}$ and $\psi_i^{*-1}(j_{\vec{t},v})$ is standard on each A_{t_i}.

3) One "stabilizes" Σ_0 by marking points on each unstable component, ie such that $2g_i + m_i < 3$. We mark $k_i = 3 - (2g_i + m_i)$ points on the smooth part of Σ_i. The total number of marked points is

$$k = \sum k_i = h^0(T\Sigma_0) = \dim \text{Aut}(\Sigma_0).$$

We obtain a stable marked curve $(\Sigma_0, \vec{p}_0)$, where $\vec{p}_0 = (p_{0,1}, \cdots, p_{0,k})$, which defines a point $[\Sigma_0, \vec{p}_0] \in \overline{\mathcal{M}}_{g,k}$. By the stability hypothesis on (Σ_0, f_0), we can do this in such a way that f_0 is an embedding on a compact neighbourhood $N = \coprod N_j$ disjoint from the nodes. In particular, we can find disjoint local transversal hypersurfaces $(H_j \subset V)$ to the images $f_0(N_j)$.

One also requires that no marked point lie in $\mathcal{U}_1$, and that $\mathcal{U}_1$ be invariant by the finite group $\mathrm{Aut}(\Sigma_0, \vec{p}_0)$, and that each $\sigma(v)$ vanishes on $\vec{p}_0$.

On $\Sigma_{\vec{t}}$ one marks $\vec{p}_{v,t} = \vec{p}_0$. This makes sense for $\vec{t} \in \Delta^m$ since these points are then in $\Sigma_0 \backslash \mathcal{U}_{\vec{t}}$ which is canonically identified with $\Sigma_{\vec{t}} \backslash \mathcal{A}_{\vec{t}}$.

Then we have the

Proposition. (cf. [LT], Lemma 3.1) *Let $\mathcal{N}$ be a small enough neighbourhood $\mathcal{N}$ of $[\Sigma_0, f_0]$ in $\overline{\mathcal{M}}_g(V, J)$. Then there is a continuous map*

$$\psi : \mathcal{N} \to \mathbf{C}^m \times \frac{H^1(T\Sigma_0)}{\mathrm{Aut}(\Sigma_0, f_0)}$$

with the following property: if $\psi(C) = (\vec{t}, [v])$ then C has a unique representative $((\Sigma_{\vec{t}}, j_{\vec{t},v}); f))$ such that

i) $f(p_{0,j}) \in H_j$ $(\forall j)$

ii) f is close to f_0 on $\Sigma_0 \backslash \mathcal{U}_1$.

This is defined as follows:

Given $C = [\Sigma, f]$ in $\overline{\mathcal{M}}_g(V, J)$ close enough to $[\Sigma_0, f_0]$, we mark $\vec{p} = (p_1, \cdots, p_k)$ on Σ such that $f(p_j) \in H_j$ $(\forall j)$; the choice of $\vec{p}$ is not unique, but this will be discussed later. We obtain a point $[\Sigma, \vec{p}]$ in the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,k}$. By the local structure of this space (cf. [HM]), it has a representative $((\Sigma_{\vec{t}}, j_{\vec{t},v}); \vec{p}_0)$ where $(\vec{t}, v)$ is small; $\vec{t}$ is uniquely defined, and v is defined up to the action of $\mathrm{Aut}(\Sigma_0, \vec{p}_0)$ on $H^1(T\Sigma_0)$. Also, it is possible to choose v such that $C = [(\Sigma_{\vec{t}}, j_{\vec{t},v}; f]$ with f satisfying i) and ii) of the proposition. Note that f is unique once $(\vec{t}, v)$ has been chosen.

There remains to identify the different possible choices for $(\vec{t}, v)$, ie find all $(\vec{t}', v')$ close to $(\vec{0}, 0)$ such that there exists $f' : \Sigma_{\vec{t}} \to V$, $(j_{\vec{t}',v'}, J)$-holomorphic and satisfying

$[\Sigma_{\vec{t}',v'}, f'] = [\Sigma_{\vec{t},v}, f]$

$f'(p_{0,j}) \in H_j$ $(\forall j)$

$f'_{|\Sigma_0 \backslash \mathcal{U}_1}$ is close to $(\vec{0}, 0, f_{0|\Sigma_0 \backslash \mathcal{U}_1})$.

We want to see that this is equivalent to $\vec{t}' = \vec{t}$, $v' = \gamma.v$ for some $\gamma \in \mathrm{Aut}(\Sigma_0, f_0)$. We shall prove only the direct implication, the converse being easy.

The above conditions are equivalent to the existence of a biholomorphism $\phi : (\Sigma_{\vec{t}}, j_{\vec{t},v}) \to (\Sigma_{\vec{t}'}, j_{\vec{t}',v'})$ such that $f' \circ \phi = f$, $\phi(p_{0,j}) \in H_j$ $(\forall j)$ and ϕ is close on $\Sigma_0 \backslash \mathcal{U}_1$ to an element $\gamma \in \mathrm{Aut}(\Sigma_0, f_0)$.

One easily proves that the subgroup $\widetilde{\Gamma}_0$ of $\mathrm{Aut}(\Sigma_0)$ generated by $\mathrm{Aut}(\Sigma_0, \vec{p}_0)$ and $\mathrm{Aut}(\Sigma_0, f_0)$ is still finite. Thus we can impose that it leaves $\mathcal{U}_1$ invariant, thus each $\gamma \in \widetilde{\Gamma}_0$ will define an isomorphism $\gamma_{\vec{t},v}$ from $(\Sigma_{\vec{t}}, j_{\vec{t},v})$ to $(\Sigma_{\vec{t}}, j_{\vec{t},\gamma.v})$.

Thus $\psi = \gamma_{\vec{t},v} \circ \phi^{-1} : \Sigma_{\vec{t}'} \to \Sigma_{\vec{t}}$ is a $(j_{\vec{t},v'}, j_{\vec{t}',\gamma.v})$-biholomorphism which is close to the identity on $\Sigma_0 \backslash \mathcal{U}_1$, and $f'(\psi(p_{0,j})) \in H_j$ $(\forall j)$. This implies $\psi(p_{0,j}) = p_{0,j}$ $(\forall j)$, thus $\vec{t} = \vec{t}'$ and $\psi = \mathrm{Id}$. Thus also $v' = \gamma.v$, which finishes the proof.

Remark. Condition iii) in the description of the topology implies that f is close to f_0 on $\Sigma \setminus \mathcal{U}_{\vec{t}}$, that is on almost all its domain of definition.

2.3. Decomposition according to the topology of the source

Denote by R_g a set of representatives for each topological type of smooth surface with nodes of arithmetic genus g. For $\Sigma_0 \in R_g$, we define $\mathcal{M}_{\Sigma_0}(V,J)$ as the subspace of $\overline{\mathcal{M}}_g(V,J)$ given by the curves $[\Sigma, f]$ with Σ diffeomorphic to Σ_0. Then by definition we have a disjoint decomposition

$$\overline{\mathcal{M}}_g(V,J) = \bigcup_{\Sigma \in R_g} \mathcal{M}_\Sigma(V,J).$$

Let us call each element of this decomposition an *(equitopological) stratum.*

One defines similarly the strata $\mathcal{M}_{\Sigma,F}(V,J)$ where (Σ, F) is a nodal surface and F a finite subset of the smooth part. Thus if $R_{g,m}$ is a set of representatives of the topological types of (Σ, F) for Σ of genus g and $|F| = m$, we have a disjoint decomposition

$$\overline{\mathcal{M}}_{g,m}(V,J) = \bigcup_{\Sigma \in R_{g,m}} \mathcal{M}_\Sigma(V,J).$$

Local structure of $\mathcal{M}_{\Sigma_0}(V,J)$

1) If Σ_0 is smooth of genus g, $\mathcal{M}_{\Sigma_0}(V,J)$ is naturally homeomorphic to the space denoted $\mathcal{M}_g(V,J)$ by [IS]. Note that this last space has, under some genericity assumption, a structure of smooth orbifold (cf. [IS] and section 4). The smooth structure comes from the fact that, up to a finite group, the space is naturally a subspace of a Banach manifold of maps.

2) More generally, assume that Σ_0 is nodal, with components $\Sigma_1, \cdots, \Sigma_r$, but restrict to the case where $\dim(V) = 4$ and f_0 is an embedding with distinct tangents near each node. Then $\mathcal{M}_\Sigma(V,J)$ is homeomorphic, and even diffeomorphic to an *open* subset in the product $\prod_{i=1}^r \mathcal{M}_{\widetilde{\Sigma}_i}(V,J)$, defined by suitable intersection properties.

RemarK; The main problem in the study of spaces of J-curves is that, although equitopological strata are well understood, the way they "hang together" in general is not.

3. Proof of Theorem 1

By 2.2, every J-curve C close to $[\Sigma_0, f_0]$ is of the form $[\Sigma_{\vec{t}}, j_{\vec{t},v}, f]$ with $\vec{t} \in \mathbf{C}^m$ well defined and $v \in H^1(T\Sigma_0)$ determined up to $\Gamma_0 = \mathrm{Aut}(\Sigma_0, f_0)$, f being well defined once v has been chosen. Also, $\vec{t}$ and v tend to 0 as $C \to C_0$, and $f \to f_0$ on the complement of the nodes (remark at the end of 2.2). Following [MS], [RT], [LT] and [FO], we shall find these curves by solving the equation for (v, f) for a fixed small $\vec{t}$:

1) First, we construct a map $w_{\vec{t}} : \Sigma_{\vec{t}} \to V$ which is approximately $(j_{\vec{t},0}, J)$-holomorphic.

2) Then we write f in the form $f = \exp_{w_{\vec{t}}}(\xi)$ where ξ belongs to $\Gamma(\Sigma_{\vec{t}}, w_{\vec{t}}^* TV \,\mathrm{rel}\, \vec{p}_0)$, the space of sections of $w_{\vec{t}}^* TV$ which are of class L_1^p for some $2 < p < +\infty$ and are tangent to H_j at each point $p_{0,j}$. The equation for (v, f) can now be written in the form $\mathcal{F}_{\vec{t}}(v, \xi) = 0$, where the target space of $\mathcal{F}_{\vec{t}}$ is $\Omega^{0,1}(w_{\vec{t}}^* TV)$, the space of forms of class L_1^p.

3) The equation is to be solved for v small and ξ small in the L^∞-norm. We prove that in fact ξ is necessarily small in any norm L^p_1. We shall then work with such a norm on the source, with the corresponding L^p-norm on the target, so that $\mathcal{F}_{\vec{t}}$ becomes a Fredholm map.

4) We prove that $\mathcal{F}_{\vec{t}}$ satisfies the hypotheses of the implicit function theorem, with some uniformity in $\vec{t}$. The key point is that its differential at zero is onto with a right inverse bounded independently of $\vec{t}$. For $\vec{t}=\vec{0}$ it is exactly the normal genericity. The existence of the inverse for $\vec{t}$ small is not obvious since the dependence of $\mathcal{F}_{\vec{t}}$ is $\vec{t}$ is not quite continuous, but the construction of [MS] can be extended to our case.

5) A delicate point is the continuity in $\vec{t}$ of the solutions, for which we use an idea of [LT].

3.1. Construction of an almost J-holomorphic map on $\Sigma_{\vec{t}}$

Let $\vec{t}\in \mathbf{C}^m$ be such that $|\vec{t}| = \max |t_i| < 1$. We construct a map $w_{\vec{t}} : \Sigma_{\vec{t}} \to V$ (corresponding to w_R in [MS]) as follows: $w_{\vec{t}} = f_0 \circ \widetilde{\rho}_{\vec{t}}$, with

$$\begin{cases} \widetilde{\rho}_{\vec{t}} = \mathrm{Id} \text{ on } \Sigma_{\vec{t}}\backslash \mathcal{A}_{\vec{t}} \\ \widetilde{\rho}_{\vec{t}}(\psi_i(x,y)) = \phi_i(\rho(|t_i|^{-1/4}|x|)\, x, 0) \text{ if } |x| \geq |y| \\ \widetilde{\rho}_{\vec{t}}(\psi_i(x,y)) = \phi_i(0, \rho(|t_i|^{-1/4}|y|)\, y) \text{ if } |x| \leq |y|. \end{cases}$$

Here $\rho : \mathbf{R}^+ \to [0,1]$ is a smooth cutoff function such that

$$\rho(s) = 0 \text{ for } s \leq 1 \text{ and } \rho(s) = 1 \text{ for } s \geq 2.$$

In particular:

$$w_{\vec{t}}(\psi_i(x,y)) = f_0 \circ \phi_i\,(0,0) \text{ if } |x| \text{ and } |y| \leq |t_i|^{1/4}.$$

Comments. We have replaced the δ of [MS], which was small but not infinitesimal, by $|t_i|^{1/4}$: this is to get the continuity in $\vec{t}$ of the constructions, notably that of the right inverse $R_{\vec{t}}$.

To measure the almost-holomorphy of $w_{\vec{t}}$, we equip $\Sigma_{\vec{t}}$ with a metric which is fixed on $\Sigma_0\backslash\mathcal{U}_1$ and induced on $\mathcal{A}_{\vec{t}}$ by the embeddings $A_{t_i} \subset \mathbf{C}^2$.

Proposition. *One has for every $2 < p < \infty$ the estimate*

$$||\overline{\partial}_{j_{0,\vec{t}}} w_{\vec{t}}||_{L^p} = O(|\vec{t}|^{1/2p}).$$

Proof. We evaluate the contribution to $||\overline{\partial}_{j_{0,\vec{t}}} w_{\vec{t}}||^p_{L^p}$ of A_{i,t_i}, or rather the half $\{|x| \geq |t_i|^{1/2}\}$. Then $\overline{\partial}_{j_{0,\vec{t}}} w_{\vec{t}}$ is $O(|t_i|^{-1/4}|x|)$ pointwise, and vanishes except if $|t_i|^{1/4} \leq |x| \leq 2|t_i|^{1/4}$. Thus this contribution is at most of the order of

$$(|t_i|^{-1/4})^p . \int_{|t_i|^{1/4}}^{2|t_i|^{1/4}} r^{p+1} dr = O\big((|t_i|^{-p/4}).(|t_i|^{p/4+1/2})\big) = O(|t_i|^{1/2}).$$

Thus the total integral is $O(|\vec{t}|^{1/2})$, which proves (ii).

3.2. Setting up the equation

We want to solve, for a given small $\vec{t}\in \mathbf{C}^m$, the equation

$$\overline{\partial}_{j_{\vec{t},v},J} f := \frac{1}{2}(df + J\, df\, j_{\vec{t},v}) = 0$$

where the unknown is (v, f) with $v \in H^1(T\Sigma_0)$ and $f \in \mathrm{Map}(\Sigma_{\vec{t}}, V)$ such that $f(p_{0,j}) \in H_j$ $(\forall j)$. Also, v and the $\mathcal{C}^0$ distance $\mathrm{dist}(f, w_{\vec{t}})$ should be small.

Choosing the metric on V so that the transversals (H_j) are totally geodesic, we can write the unkown f uniquely in the form

$$f = \exp_{w_{\vec{t}}}(\xi)\ ,\ \xi \in \Gamma(w_{\vec{t}}^* TV \mathrm{rel}\, \vec{p}_0) \text{ small in the } L^\infty \text{ norm.}$$

The expression $\overline{\partial}_{j_{\vec{t},v}} f$ takes values in $\Omega^{0,1}(f^*TV)$. Let us fix a J-complex connection on TV and use it to define the parallel transport along geodesics. We then get an isomorphism $\Pi_{\vec{t}}$ from $\Omega^{0,1}(f^*TV)$ to $\Omega^{0,1}(w_{\vec{t}}^*TV)$. Thus we have to solve the equation $\mathcal{F}_{\vec{t}}(v, \xi) = 0$, where

$$\mathcal{F}_{\vec{t}} : H^1(T\Sigma_0) \times \Gamma(w_{\vec{t}}^* TV \mathrm{rel}\, \vec{p}_0) \to \Omega^{0,1}(w_{\vec{t}}^*TV)$$

is defined by

$$\mathcal{F}_{\vec{t}}(v, \xi) = \Pi_{\vec{t}}(\overline{\partial}_{j_{\vec{t},v}, J}(\exp_{w_{\vec{t}}}(\xi))).$$

More precisely, we want to describe all the solutions such that $|v|$ and $||\xi||_{L^\infty}$ are small.

The map $\mathcal{F}_{\vec{t}}$ is defined on a ball $B(\epsilon_0) = \{(v, \xi) \mid ||(v, \xi)|| < \epsilon_0\}$ independent of $\vec{t}$, and satisfies

$$||\mathcal{F}_{\vec{t}}(0, 0)|| = O(|\vec{t}|^{1/2p}).$$

Another property of $\mathcal{F}_{\vec{t}}$ is that it is of class $\mathcal{C}^\infty$. To see this, note that its pointwise value can be written

$$(\mathcal{F}_{\vec{t}}(v, \xi))\,(z) = A(w_{\vec{t}}(z), v, \xi(z)).\nabla w_{\vec{t}}(z) + B(w_{\vec{t}}(z), v, \xi(z)).\nabla \xi(z),$$

where A and B are $\mathcal{C}^\infty$. This also implies that for every $k \in \mathbf{N}$, $||\mathcal{F}_{\vec{t}}||_{\mathcal{C}^k(B(\epsilon_0))}$ is bounded, with a bound independent of $\vec{t}$.

3.3. Smallness of ξ in the L_1^p-norm

Proposition. *For every $\epsilon > 0$ there exists $\eta > 0$ with the following property. Let $f = \exp_{w_{\vec{t}}}(\xi)$ be $(j_{v,\vec{t}}, J)$-holomorphic, with $|v|$, $|\vec{t}|$ and $||\xi||_{L^\infty}$ less than η. Then $||\xi||_{L_1^p}$ is less than ϵ.*

Proof. Let U be any neighbourhood of the nodes. By the elliptic apriori estimates, we can find η such that $||\xi_{|\Sigma_0 \setminus U}||_{L_1^p}$ is less than $\frac{\epsilon}{2}$. Thus the proposition is reduced to a local property near the nodes.

Let us work in a local chart $(V, v) \approx (\mathbf{C}^n, 0)$ near the image of a node, J-holomorphic at v. Then the holomorphy equation for f can be written $\overline{\partial} w + q(w).\partial w = 0$ where q is a map from $\mathbf{C}^n$ to $\overline{\mathrm{End}}_{\mathbf{C}}(\mathbf{C}^n)$. Also, replacing the coordinate w by δw with δ arbitrarily small, we can assume that q is arbitrarily small in the $\mathcal{C}^1$-topology.

Using a Riemannian metric on V which is a multiple of the standard one in this chart, the proposition is reduced to the following lemma.

Lemma. For every ϵ there exists η with the following property. Let q be a map from the polydisk $\Delta^n \subset \mathbf{C}^n$ to $\overline{\mathrm{End}}_{\mathbf{C}}(\mathbf{C}^n)$ and let w and w' be maps from A_t to Δ^n which satisfy the equations

$$\overline{\partial} w + q(w).\partial w = 0 = \overline{\partial} w' + q(w').\partial w' = 0$$

and the inequalities

$$\max\left(|t|, ||q||_{\mathcal{C}^1}, ||w||_{L_1^p}, ||w' - w||_{L^\infty}\right) < \eta.$$

Then $||w'-w||_{L^p_1(A'_t)} < \epsilon$, where A'_t is the intersection of A_t with $\Delta_{1/2} \times \Delta_{1/2}$.

Proof of the lemma

We first construct a right inverse P_t to $\overline{\partial} : L^p_1(A_t) \to L^p\Omega^{0,1}(A_t)$ which is uniformly bounded. An element of $L^p\Omega^{0,1}(A_t)$ can be decomposed in two parts, one with support in the subannulus $A_t^+ = \{|x| \geq |y|\}$, the other in the subannulus $A_t^- = \{|y| \geq |x|\}$. It suffices to solve $\overline{\partial} f = \alpha$ for α with support in A_t^+.

We write $\alpha = g(x)\overline{dx}$ where g is a function of class L^p defined on $\Delta \setminus \Delta_{|t|^{1/2}}$, and we define

$$P_t(\alpha)(x,y) = Pg(x)$$

using the standard inverse

$$Pg(x) = -\frac{1}{\pi}\int_\Delta \frac{g(z)}{z-x}\, d\sigma_z,\ \ x \in \mathbf{C}.$$

Recall that $\overline{\partial}(Pg) = g$ on $\Delta \setminus \mathrm{Supp}(g)$ and 0 elsewhere. By Vekua (or Calderon-Zygmund), P is bounded from $L^p(\Delta)$ to $L^p_1(\Delta)$. Since the projection $(x,y) \mapsto x$ from A_t to $\mathbf{C}$ is Lipschitz, this implies that P_t is uniformly bounded.

Let $\xi = w' - w$, so that $||\xi||_{L^\infty}$ is arbitrarily small. Taking the difference between the equations and developing $q(w+\xi) - q(w)$ to the first order, we get the linear equation

$$\overline{\partial}\xi + q.\partial\xi + a.\xi = 0$$

where we now consider q, as well as a, as a map defined on A_t. These maps are arbitrarily small in the L^∞-norm.

We want to prove that then a solution of this equation satisfies $||q||_{L^p_1} \leq C||q||_{L^\infty}$ with C independent of t. This can be done in a way similar to [Sik1] p.171-172 which treats the case of a disk. One uses a smooth cutoff function $\rho : \Delta^2 \to [0,1]$, which vanishes near the boundary and equals 1 on $\Delta_{1/2} \times \Delta_{1/2}$. Then setting $\xi_1 = \rho\xi$ one obtains

$$\overline{\partial}\xi_1 + q.\partial\xi_1 = -(\overline{\partial}\rho + q.\partial\rho).\xi - a.\xi_1 = g.$$

The right-hand side g is $O(||\xi||)$ in the L^∞-norm. It suffices to bound $||\xi||_{L^p_1}$ by $||\xi||_{L^\infty}$, ie to bound $||\overline{\partial}\xi||_{L^p}$ and $||\partial\xi||_{L^p}$.

Write $\xi_1 = P_t(\overline{\partial}\xi_1) + h$ where h is holomorphic, and set $T_t = \partial P_t$ so that T_t is uniformly bounded in L^p and $\partial\xi_1 = T_t(\overline{\partial}\xi_1) + \partial h$.

Note that the definition of P_t implies that $||P_t(\overline{\partial}\xi_1)||_{L^2} \leq ||\xi_1||_{L^\infty}$, thus $||h||_{L^2} \leq C||\xi||_{L^\infty}$. Since h is holomorphic, one has $||h||_{L^\infty(A'_t)} \leq C||h||_{L^2(A'_t)}$ and a fortiori $||h||_{L^p(A'_t)} \leq C||h||_{L^2(A'_t)}$ Thus

$$\begin{aligned} ||\partial\xi_1||_{L^p(A_t)} = ||\partial\xi_1||_{L^p(A'_t)} &\leq C(||\overline{\partial}\xi_1||_{L^p(A_t)} + ||h||_{L^p(A'_t)}) \\ &\leq C(||\overline{\partial}\xi_1||_{L^p} + C||\xi||_{L^\infty}). \end{aligned}$$

The equation for ξ implies then $||\overline{\partial}\xi_1||_{L^p}(1 - C\eta) \leq C'||\xi||_{L^\infty}$. Thus for η small enough we bound $||\overline{\partial}\xi_1||_{L^p}$ by $||\xi||_{L^\infty}$, and then also $||\partial\xi_1||_{L^p}$ via the estimate above. This concludes the proof of the lemma and thus of the proposition.

3.4. Fredholm property and surjectivity of $d\mathcal{F}_{\vec{0}}(0,0)$

Proposition.

i) The operator $D_{\vec{t}} = d\mathcal{F}_{\vec{t}}(0,0)$ is Fredholm of index $i(A,g) - m$.

ii) The cokernel of $D_{\vec{0}}$ is identified with that of $\widetilde{D}_{f_0}$. In particular, $D_{\vec{0}}$ is onto if and only if $[\Sigma_0, f_0]$ is normally generic.

Proof. First of all, let us observe that the linearization operators D_{f_0} and $\widetilde{D}_{f_0}$ can be defined when one replaces f_0 with any map w of class L^p_1, giving operators D_w and $\widetilde{D}_w = \frac{1}{2} J\, dw \oplus D_w$ with the same properties.

i) The operator $D_{\vec{t}}$ is the restriction of $\widetilde{D}_{w_{\vec{t}}}$ to $\sigma(H^1(T\Sigma_0)) \times \Gamma(w^*_{\vec{t}} TV \mathrm{rel}\, \vec{p}_0)$. Since $D_{w_{\vec{t}}}$ is Fredholm and $H^1(T\Sigma_0)$ has finite dimension, this proves that $D_{\vec{0}}$ is Fredholm. Its index is

$$\begin{aligned}\mathrm{ind}(D_{\vec{0}}) &= h^1(T\Sigma_0) + \mathrm{ind}(D_{f_0} \mathrm{rel}\, \vec{p}_0)\\ &= h^1(T\Sigma_0) + \mathrm{ind}(D_{f_0}) - h^0(T\Sigma_0)\\ &= \mathrm{ind}(D_{f_0}) - \chi(T\Sigma_0).\end{aligned}$$

On the other hand,

$$\begin{aligned}\chi(T\Sigma_0) &= \chi(T\widetilde{\Sigma}_0) - 2m = \sum (3 - 3g_i) - 2m\\ &= 3 - 3g + 3m - 2m = 3 - 3g + m,\end{aligned}$$

thus $\mathrm{ind}(D_{\vec{t}}) = \mathrm{ind}(D_{f_0}) + 3g - 3 - m$, qed.

ii) The operator $D_{\vec{0}}$ is the restriction of $\widetilde{D}_{f_0} = \frac{1}{2} J\, df_0 \oplus D_{f_0}$ to $\sigma(H^1(T\Sigma_0)) \times \Gamma(f_0^* TV \mathrm{rel}\, \vec{p}_0)$. We want to prove that this does not diminish the image, ie that

- restricting from $\Omega^{0,1}(T\Sigma_0)$ to $\sigma(H^1(T\Sigma_0))$: write $\widetilde{v} = \sigma(v) + \overline{\partial} u$ with $u \in \Gamma(T\Sigma_0)$, thus $D_{\vec{0}}(\widetilde{v}, 0) = D_{\vec{0}}(\sigma(v), \frac{1}{2} j u)$

- restricting from $\Gamma(f_0^* TV)$ to $\Gamma(f_0^* TV \mathrm{rel}\, \vec{p}_0)$: there exists $X \in H^0(T\Sigma_0)$ such that $\xi(p_j) - df_0(X(p_j)) \in TH_j$, thus $D_{\vec{0}}(0, \xi) = D_{\vec{0}}(0, \xi - df_0\, X)$.

3.5. Existence of a uniformly bounded right inverse for $D_{\vec{t}}$

Proposition. *For $|\vec{t}|$ small enough, $D_{\vec{t}}$ has a right inverse which is uniformly bounded.*

Proof. One constructs a quasi-inverse $Q_{\vec{t}}(\eta)$ as in [MS], ie an operator which satisfies

$$||D_{\vec{t}} \circ Q_{\vec{t}} - \mathrm{Id}||_{L^p} = o(1) \ , \ \vec{t} \to \vec{0}.$$

For this, we shall need another almost J-holomorphic map $u_{\vec{t}}$, corresponding to (u_R, v_R) of [MS], this time defined on Σ_0: $u_{\vec{t}} = f_0 \circ \widehat{\rho}_{\vec{t}}$, with

$$\begin{cases} \widehat{\rho}_{\vec{t}} = \mathrm{Id} \text{ on } \Sigma_0 \backslash \mathcal{U}_{\vec{t}} \\ \widehat{\rho}_{\vec{t}}(\phi_i(x,0)) = \phi_i(\rho(|t_i|^{-1/4}|x|)\, x, 0) \text{ if } |t|^{1/2} \leq |x| \leq 1 \\ \widehat{\rho}_{\vec{t}}(\phi_i(0,y)) = \phi_i(0, \rho(|t_i|^{-1/4}|y|)\, y) \text{ if } |t|^{1/2} \leq |y| \leq 1. \end{cases}$$

To compare $u_{\vec{t}}$ with $w_{\vec{t}}$, define $\Gamma_{\vec{t}}$ to be the union of the circles (or points)

$$\Gamma_i = \psi_i(\{(x,y) \in \mathbf{C}^2 \mid xy = t_i \text{ and } |x| = |y| = |t_i|^{1/2})\},$$

and $\Gamma_{\vec{t}}$ the union of the Γ_i. Let $\pi_{\vec{t}} : \Sigma_{\vec{t}} \backslash \Gamma_{\vec{t}} \to \Sigma_0$ be the map which is the identity on $\Sigma_{\vec{t}} \backslash \mathcal{A}_{\vec{t}}$, and satisfies

$$\pi_{\vec{t}} \circ \psi_i(x,y) = \phi_i(x,0) \text{ or } \phi_i(0,y)$$

depending whether $|x| > |y|$ or $|x| < |y|$. It is a biholomorphism onto its image $\Sigma_0 \backslash \mathcal{D}_{\vec{t}}$.

Then we have

$$w_{\vec{t}} = \begin{cases} u_{\vec{t}} \circ \pi_{\vec{t}} \text{ on } \Sigma_{\vec{t}} \backslash \Gamma_{\vec{t}} \\ \phi_i(0,0) \text{ on } \Gamma_i \end{cases}$$

Replacing $w_{\vec{t}}$ by $u_{\vec{t}}$ in the definition of $\mathcal{F}_{\vec{t}}$, one obtains a map

$$\mathcal{G}_{\vec{t}} : \Omega^{0,1}(u_{\vec{t}}^* TV) \to \Gamma(u_{\vec{t}}^* TV).$$

We have $\mathcal{G}_{\vec{0}} = \mathcal{F}_{\vec{0}}$, thus $d\mathcal{G}_{\vec{0}}(0,0) = d\mathcal{F}_{\vec{0}}(0,0)$ is onto. It is easy to see that $u_{\vec{t}}$ converges to f_0 in the L_1^p topology as $\vec{t} \to \vec{0}$. Thus $d\mathcal{G}_{\vec{t}}(0,0)$ has a right inverse $\widetilde{R}_{\vec{t}}$ for $|\vec{t}|$ small enough, which is uniformly bounded and is continuous in $\vec{t}$.

The operator $Q_{\vec{t}}$ is defined as the composition of the three following maps:

$$\Omega^{0,1}(w_{\vec{t}}^* TV) \xrightarrow{\pi_{\vec{t}}^{*-1} \cup 0} \Omega^{0,1}(u_{\vec{t}}^* TV) \xrightarrow{\widetilde{R}_{\vec{t}}} H \times \Gamma(u_{\vec{t}}^* TV) \xrightarrow{\mathrm{Id} \times E_{\vec{t}}} H \times \Gamma(w_{\vec{t}}^* TV)$$

The first map is $\pi_{\vec{t}}^{*-1}$ followed by the extension by zero from $\Sigma_0 \backslash \mathcal{D}_{\vec{t}}$ to Σ_0. The operator $E_{\vec{t}}$ is defined by

$$(E_{\vec{t}}\xi)(z) = \begin{cases} \xi(z) & \text{if } z \notin \mathcal{A}_{\vec{t}} \\ \xi\left(\phi_i(x,0)\right) + \beta_{|t_i|}(x)\left(\xi\left(\phi_i(0,y)\right) - \xi\left(z_i\right)\right) & \text{if } z = \psi_i(x,y), |x| \geq |y| \\ \xi\left(\phi_i(0,y)\right) + \beta_{|t_i|}(y)\left(\xi\left(\phi_i(x,0)\right) - \xi\left(z_i\right)\right) & \text{if } z = \psi_i(x,y), |y| \geq |x|. \end{cases}$$

Here $\beta_\delta(z) = \rho(\frac{4 \log |z|}{\log \delta})$, where $\rho : \mathbf{R} \to [0,1]$ is the preceding cutoff function, satisfying $\rho(s) = 0$ for $s \leq 1$, $\rho(s) = 1$ for $s \geq 2$. One easily shows the following properties:

$$\beta_\delta(z) = 1 \text{ for } |z| \leq \delta^{1/2} \ , \ \beta_\delta(z) = 0 \text{ for } |z| \geq \delta^{1/4}$$
$$\int_{\mathbf{C}} |\nabla(\beta_\delta(z))|^p \, |z|^{p-2} d\sigma_z = o(1) \text{ for } \delta \to 0.$$

Note that the cutoff takes place at points $\psi_i(x,y)$ with $|t_i|^{1/2} \leq |x|, |y| \leq |t_i|^{1/4}$, where $w_{\vec{t}}$ is equal to the constant $f_0 \circ \phi_i(0,0)$.

Clearly, $E_{\vec{t}}$ is uniformly bounded for $\vec{t} \to \vec{0}$, and thus the same is true for $Q_{\vec{t}}$.

Lemma. *One has the estimate*

$$||D_{w_{\vec{t}}}(E_{\vec{t}}\xi) - D_{u_{\vec{t}}}\xi||_{L^p} = o(||\xi||_{L_1^p}) \ , \ \vec{t} \to \vec{0}.$$

Proof. We need to estimate

$$\alpha = D_{w_{\vec{t}}}(E_{\vec{t}}\xi) - D_{u_{\vec{t}}}\xi.$$

This form vanishes except at points $z = \psi_i(x,y)$ with $|x|, |y| \leq |t_i|^{1/4}$. We can assume $|x| \geq |y|$, thus $|t_i|^{1/2} \leq |x| \leq |t_i|^{1/4}$. Since $w_{\vec{t}}(z) = u_{\vec{t}}(\phi_i(x,0)) = v_i$, $D_{w_{\vec{t}}}$ and $D_{u_{\vec{t}}}$ are both equal to the usual $\overline{\partial}$-operator for maps into the complex vector space $T_{v_i}V$, thus

$$\alpha_z = \overline{\partial}(E_{\vec{t}}\xi)\left(\psi_i(z)\right) - \overline{\partial}\xi(\phi_i(x,0)).$$

Also, $\overline{\partial}\xi(0,y) = \pi_{\vec{t}}^{*-1}\eta(0,y) = 0$ since $|y| \leq |t_i|^{1/2}$, thus

$$\alpha_z = -\overline{\partial}\big(\beta_{|t_i|}(x)\big)\,.\,\big(\xi(\phi_i(0,t_i x^{-1})) - \xi(z_i)\big).$$

Since the metric on A_{t_i} is here equivalent to $|dx|^2$, we can work in the coordinate x. Using the inequality $|t_i x^{-1}| \leq |x|$ and the continuous injection $L^p_1 \to \mathcal{C}^{1-2/p}$, we have the pointwise estimate

$$|\alpha_z| \leq \text{const.}||\xi||_{L^p_1}.|\overline{\partial}\big(\beta_{|t_i|^{1/4}}(x)\big)|.|x|^{1-2/p}$$

Thanks to the integral inequality on $\nabla\beta_\delta$, we obtain the lemma.

We can now prove the quasi-inverse property. Let $\alpha = D_{\vec{t}} \circ Q_{\vec{t}}(\eta) - \eta$ for a given $\eta \in \Omega^{0,1}(w_{\vec{t}}^* TV)$. Setting $(v,\xi) = \widetilde{R}_{\vec{t}} \circ \pi_{\vec{t}}^{*-1}(\eta)$, we have $\alpha_z = df_0.v + D_{w_{\vec{t}}}(E_{\vec{t}}\xi)$ with $df_0.v + D_{u_{\vec{t}}}\xi = 0$. Thus $\alpha = D_{w_{\vec{t}}}(E_{\vec{t}}\xi) - D_{u_{\vec{t}}}\xi$, and the lemma gives $||\alpha||_{L^p} = o(||\xi||_{L^p_1})$ as $\vec{t} \to \vec{0}$. Since $||\xi||_{L^p_1} = O(||\eta||_{L^p})$, this proves $||D_{\vec{t}} \circ Q_{\vec{t}} - \mathrm{Id}|| = o(1)$ as $\vec{t} \to \vec{0}$.

3.6. Isomorphism from $\ker D_{\vec{t}}$ to $\ker D_{\vec{0}}$

Proposition. *Let $\chi_{\vec{t}} : H^1(T\Sigma_0) \times \Gamma(w_{\vec{t}}^* TV) \to \ker D_{\vec{0}}$ be defined by sending (v,ξ) to (v_0,ξ_0) which minimizes*

$$|v_0 - v|^2 + \int_{\Sigma_0 \backslash \mathcal{U}_1} |\xi_0 - \xi|^2 \, d\sigma.$$

Then the restriction of $\chi_{\vec{t}}$ to $\ker D_{\vec{t}}$ is a linear isomorphism for $|\vec{t}|$ small enough, and moreover there is a uniform estimate

$$||\chi_{\vec{t}}(v,\xi)||_{L^2} \geq C^{-1}||(v,\xi)||_{L^p_1}.$$

Proof. Arguing by contradiction, assume that there exists $\vec{t_n} \to \vec{0}$ and $(v_n,\xi_n) \in \ker D_{\vec{t_n}}$ such that $||(v_n,\xi_n)||_{L^p_1} = 1$ and $||\chi_{\vec{t_n}}(v_n,\xi_n)||_{L^2} = o(1)$. Denote $\chi_{\vec{t_n}}(v_n,\xi_n) = (v_n', \xi_n')$. By definition, $(v_n - v_n', \xi_n - \xi_n')$ is orthogonal to $\ker D_{\vec{t_n}}$ on $\Sigma_0 \backslash \mathcal{U}_1$.

There is a subsequence (v_n,ξ_n) which converges away from the nodes to $(v,\xi) \in \ker D_{\vec{0}}$. Since (v_n',ξ_n') converges to 0 in $L^2(\Sigma_0 \backslash \mathcal{U}_1)$, (v,ξ) is orthogonal to $\ker D_{\vec{0}}$ on $\Sigma_0 \backslash \mathcal{U}_1$, thus $(v,\xi) = (0,0)$. This implies that $|v_n| = o(1)$ and $\xi_n \to 0$ away from the nodes, uniformly in the $\mathcal{C}^\infty$ topology. Also $D_{w_{\vec{t_n}}}\xi_n = 0$ on each annulus $A_{t_{n,i}}$.

On each annulus $A_t = A_{t_{n,i}}$ this equation takes the form

$$\overline{\partial}\xi + q(w).\partial\xi + (dq(w).\xi).\partial w = 0$$

with $||q||_{L^\infty}$ and $||dq||_{L^\infty}$ are arbitrarily small. Also, $||w||_{L^\infty}$ and $||dw||_{L^\infty}$ are bounded, and is ξ is arbitrarily $\mathcal{C}^\infty$ small away from $(0,0)$. In particular $||\xi_{|\partial A_{t_i}}||_{L^\infty}$ is arbitrarily small.

We shall get a contradiction via the lemma in 3.3, provided we can prove a "weak maximum principle", namely the inequality

$$||\xi||_{L^\infty} \leq C||\xi_{|\partial A_{t_i}}||_{L^\infty}.$$

To this effect, we set $\widetilde{\xi} = \xi + q(w).\xi$ (compare [Sik2], proof of Proposition 1). We have

$$\overline{\partial}\widetilde{\xi} = -(\overline{\partial}(q(w)).\xi - (dq(w).\xi).\partial w = A.\widetilde{\xi},$$

where A is complex-linear and is $o(1)$ in the L^∞-norm.

As in [Sik2], the fact that P_t is onto with a bounded right-inverse $L^p \to L^p_1$ gives a solution $\Phi : A_t \to \mathrm{G}\ell(\mathbf{C}^n)$ of the resolvant equation $\overline{\partial}\Phi = A.\Phi$. This solution lives in L^p_1 and satisfies $\max(||\Phi||_{L^\infty}, ||\Phi^{-1}||_{L^\infty}) \leq 2$ for $||A||_{L^\infty}$ small enough. Then writing $\widetilde{\xi} = \Phi.h$, we have that h is holomorphic, thus $||h||$ satisfies the usual maximum principle on A_t. Thus the weak maximum principle is satisfied with $C = 4$, which finishes the proof of the proposition.

3.7. End of the proof of Theorem 1

We have proved that the map

$$\mathcal{F}_{\vec{t}} : H^1(T\Sigma_0) \times \Gamma(w_{\vec{t}}^* TV \operatorname{rel} \vec{p}_0) \to \Omega^{0,1}(w_{\vec{t}}^* TV),$$

defined for $|\vec{t}| < 1$ on the ball $\{|(v,\xi)| < \epsilon_0\}$, has the following properties:

$$\begin{cases} \mathcal{F}_{\vec{t}} \text{ is of class } \mathcal{C}^2 \text{ and } ||\mathcal{F}_{\vec{t}}||_{\mathcal{C}^2} \leq C_1 \\ |\mathcal{F}_{\vec{t}}(0,0)| \leq C_2 |t|^{1/2p} \\ \text{for } |\vec{t}| < \epsilon_1,\ D_{\vec{0}} \text{ has a right inverse } R_{\vec{t}} \text{ such that } |R_{\vec{t}}| \leq C_3. \end{cases}$$

Denote

$$K_{\vec{t}} = \ker D_{\vec{0}}\ ,\ K_{\vec{t}}(\epsilon) = \{(v,\xi) \in K_{\vec{t}} \mid |(v,\xi)| < \varepsilon\}.$$

By the implicit function theorem, we obtain a map

$$\Psi_{\vec{t}} = (\nu_{\vec{t}}, \Phi_{\vec{t}}) : K_{\vec{t}}(\epsilon) \to H^1(T\Sigma_0) \times \Gamma(w_{\vec{t}}^* TV \operatorname{rel} \vec{p}_0)$$

defined for $|\vec{t}|$ and ϵ small enough, with the following property. If we set

$$f_{\vec{t},v,\xi} = \exp_{w_{\vec{t}}}(\Phi_{\vec{t}}(v,\xi)) : \Sigma_{\vec{t}} \to V,$$

then

$$\begin{cases} f_{\vec{t},v,\xi} \ is \ (j_{\vec{t},\nu_{\vec{t}}(\vec{t},v)}, J)\text{-holomorphic on } \Sigma_{\vec{t}} \\ f_{\vec{t},v,\xi}(p_{0,j}) \in H_j \ (\forall i) \\ \operatorname{dist}(f_{\vec{t},v,\xi}, w_{\vec{t}}) < \varepsilon. \end{cases}$$

Thus $[\Sigma_{\vec{t},\nu_{\vec{t}}(v,\xi)}, f_{\vec{t},v,\xi}]$ is an element of the neigbourhood $\mathcal{N}_\epsilon$ of $[\Sigma_0, f_0]$

Conversely, paragraph 3.3 and the uniqueness in the implicit function theorem imply that, for some $\epsilon_1 < \epsilon$ and for every $\vec{t}$ such that $|\vec{t}| < \epsilon_1$, the following property holds: for every map h defined on $\Sigma_{\vec{t}}$

$$\begin{cases} h \text{ is } (j_{v'}, J)\text{-holomorphic on } \Sigma_{\vec{t}} \\ h(p_{0,j}) \in H_j \ (\forall i) \\ \operatorname{dist}(h, w_{\vec{t}}) < \epsilon_1 \end{cases}$$

there exists $(v,\xi) \in K_{\vec{t}}(\epsilon)$ such that $h = f_{\vec{t},v,\xi}$ and $v' = \nu_{\vec{t}}(v,\xi)$.

Let us now define $\phi_J : \mathcal{N} \to \mathbf{C}^m \times \frac{\ker D_{\vec{0}}}{\mathrm{Aut}(\Sigma_0, f_0)}$. We start with $\Phi(C) = (\vec{t}, [(v, f)])$. We write $f = \exp_{w_{\vec{t}}} \xi$, and we project (v, ξ) orthogonally on $(v_0, \xi_0) \in \ker D_{\vec{0}}$, ie $(v_0, \xi_0) = \chi_{\vec{t}}.(v, \xi)$. This is well defined up to the action of $\mathrm{Aut}(\Sigma_0, f_0)$. Then we set

$$\phi_J(C) = (\vec{t}, [v_0, \xi_0]).$$

This is clearly continuous, there remains to see that it is locally bijective.

This finishes the proof of Theorem 1 for J fixed. When J varies, it suffices to apply the parametric version of the implicit function theorem.

4. Normal genericity and the normal $\overline{\partial}$-operator in dimension 4

In this section we define the operator D^{N_0} presented in the Introduction. It has been defined by [IS] for all curves parameterized by a smooth surface Σ_0, we shall need to extend the definition when Σ_0 has nodes. We shall do it only in dimension 4, since in higher dimension there are some complications and anyway it is probably not very useful. Thus we assume in this section that (V, J) is a 4-dimensional almost-complex manifold.

4.1. The operator D^{N_0}

Let Σ_0 be a nodal Riemann surface, and let $f_0 : \Sigma_0 \to V$ be a J-holomorphic map, with no constant component.

1) If Σ_0 is smooth, following [IS] one defines the line bundle $N_0 = f_0^* TV / \overline{df_0(T\Sigma_0)}$, and $D^{N_0} : \Gamma(N_0) \to \Omega^{0,1}(N_0)$ is induced by D_{f_0}. It is proved in [IS] (paragraphs 1.3 and 1.5, see also 5[B], paragraph 1.1) that D^{N_0} is an operator of type $\overline{\partial} + a$ defined from L^p_1 sections to L^p forms, and that one has an exact diagram

$$\begin{array}{ccccccc} 0 \to & \Gamma(T\Sigma_0 \otimes L) & \longrightarrow & \Gamma(f_0^* TV) & \longrightarrow & \Gamma(N_0) & \to 0 \\ & \overline{\partial} \downarrow & & D_{f_0} \downarrow & & D^{N_0} \downarrow & \\ 0 \to & \Omega^{0,1}(T\Sigma_0 \otimes L) & \longrightarrow & \Omega^{0,1}(f_0^* TV) & \longrightarrow & \Omega^{0,1}(N_0) & \to 0 \end{array} \tag{4.1}$$

Here $L = L(df_0^{-1}(0))$ is the divisor of zeros of df_0, counted with multiplicities. In particular, if f_0 is an immersion:

$$\begin{array}{ccccccc} 0 \to & \Gamma(T\Sigma_0) & \longrightarrow & \Gamma(f_0^* TV) & \longrightarrow & \Gamma(N_0) & \to 0 \\ & \overline{\partial} \downarrow & & D_{f_0} \downarrow & & D^{N_0} \downarrow & \\ 0 \to & \Omega^{0,1}(T\Sigma_0) & \longrightarrow & \Omega^{0,1}(f_0^* TV) & \longrightarrow & \Omega^{0,1}(N_0) & \to 0 \end{array}.$$

2) If Σ_0 has nodes, we make the following assumption:

(*) *f_0 is an embedding near each node, with distinct tangents.*

Consider the normalization $\nu : \widetilde{\Sigma}_0 \to \Sigma_0$ and the induced map $\widetilde{f}_0 = f_0 \circ \nu$. One can associate as in 1) the normal bundle $\widetilde{N}_0$ over $\widetilde{\Sigma}_0 = \coprod_{i=1}^r \widetilde{\Sigma}_i$ and the operator $D^{\widetilde{N}_0}$. By definition, we set $N_0 = \widetilde{N}_0$ and $D^{N_0} = D^{\widetilde{N}_0}$.

Let us prove that the diagram (4.1) "over $\widetilde{\Sigma}_0$" remains exact "over Σ_0". Only the first line is changed: $\Gamma(T\Sigma_0 \otimes L)$ (resp. $\Gamma(f_0^* TV)$) can be identified with elements of $\Gamma(\widetilde{T\Sigma_0} \otimes L)$ (resp.

$\Gamma(\widetilde{f}_0^*TV))$ vanishing on the inverse images z^+ and z^- of any node (resp. having the same value on z^+ and z^-).

The problem is to prove that $\Gamma(f_0^*TV) \to \Gamma(N_0)$ is still onto. This follows immediately from the fact that, for each double point $v = \widetilde{f}_0(z^+) = \widetilde{f}_0(z^-)$, the natural map

$$T_vV \to \frac{T_vV}{d\widetilde{f}_0(T_{z^+})} \oplus \frac{TvV}{d\widetilde{f}_0(T_{z^-})}$$

is onto by the assumption (*).

Remark. This is where we use the dimension 4: in higher dimension, we would have to restrict $D^{\widetilde{N}_0}$ to some suitable subspace of $\Gamma(\widetilde{N}_0)$.

4.2. Index computation

Using the exact diagram (4.1) and the index formula for D_{f_0} p.2, one gets the following result.

Proposition. *One has*

$$\operatorname{ind}(D^{N_0}) = \langle c_1(TV), A\rangle + g - 1 - m - |df_0^{-1}(0)|.$$

Other proof: applying Riemann-Roch separately to each component, one has $\operatorname{ind}(D^{N_0}) = \sum_i (c_1(\widetilde{N}_i) + 1 - g_i)$. The short exact sequence $T\widetilde{\Sigma}_i \to \widetilde{f}_0^*TV \to \widetilde{N}_i$ gives

$$c_1(\widetilde{N}_i) = \langle c_1(TV), A_i\rangle - 2(1 - g_i) - |df_i^{-1}(0)|.$$

Using the fact that $\sum (1 - g_i) = 1 - g + m$, we get the proposition.

Remark. Note that $\langle c_1(TV), A\rangle + g - 1 = i(A, g)$.

4.3. Normal genericity and surjectivity of D^{N_0}

Let $f_0 : \Sigma_0 \to V$ be a J-holomorphic map such that D^{N_0} can be defined. Then the exact diagram (4.1) implies the isomorphism

$$\frac{\Omega^{0,1}(f_0^*TV)}{df_0(\Omega^{0,1}(T\Sigma_0 \otimes L) \oplus D_{f_0}(\Gamma(f_0^*TV))} \simeq \operatorname{coker} D^{N_0}.$$

Since $\widetilde{D}_{f_0} = \frac{1}{2}J\, df_0 \oplus D_{f_0}$ and $J\, df_0 = df_0\, j_0$, one has

$$\operatorname{coker} \widetilde{D}_{f_0} = \frac{\Omega^{0,1}(f_0^*TV)}{df_0(\Omega^{0,1}(T\Sigma_0)) \oplus D_{f_0}(\Gamma(f_0^*TV))}.$$

Since $\Omega^{0,1}(T\Sigma_0) \subset \Omega^{0,1}(T\Sigma_0 \otimes L)$, there is a natural embedding from $\operatorname{coker} D^{N_0}$ to $\operatorname{coker} \widetilde{D}_{f_0}$. Furthermore, the induced map $H^{0,1}(T\Sigma_0) \to \Omega^{0,1}(T\Sigma_0 \otimes L)$ is onto, thus if $\alpha \in \Omega^{0,1}(T\Sigma_0 \otimes L)$ there exist $\beta \in \Omega^{0,1}(T\Sigma_0)$ and $\xi \in \Gamma(T\Sigma_0 \otimes L)$ with $\alpha = \beta + \overline{\partial}\xi$, so that $df_0(\alpha) = df_0(\beta) + D_{f_0}\, df_0(\xi)$ belongs to $\operatorname{im} \widetilde{D}_{f_0}$.

Thus $\operatorname{coker} D^{N_0} = \operatorname{coker} \widetilde{D}_{f_0}$, thus normal genericity is equivalent to the surjectivity of D^{N_0}.

5. The case of dimension 4

5.1. Adjunction formula

The most important special property for J-curves in dimension 4 is the "positivity of intersections" and in particular its corollary the "adjunction formula" (cf. [McD], [MW], [Sik2]). We state here a version allowing the source of the curves to have nodes.

Proposition. *Let $f : \Sigma \to (V^4, J)$ be a simple J-holomorphic map, where Σ is a connected Riemann surface with m nodes. Then to each singularity s of the image $S = f(\Sigma)$ one can associate a strictly positive integer $\delta(s)$, with the property that*

$$\sum_{s \in \mathrm{Sing}(S)} \delta(s) - m = g_a(A) - g_a(\Sigma).$$

Here $g_a(A)$ is the arithmetic genus of A, which is equal to $\frac{1}{2}(A.A - \langle c_1(TV), A\rangle) + 1$.

Note that each node z contributes at least 1 to the sum on the left, and exactly 1 if and only if it is embedded with distinct tangents and $f^{-1}(f(z)) = \{z\}$.

5.2. Nodal curves

Let (V, J) be an almost complex manifold of dimension 4. We say that a J-holomorphic curve $C_0 = [\Sigma_0, f_0]$ is *nodal* if f_0 is an embedding. By 5.1, the arithmetic genera $g_a(\Sigma)$ and $g_a(A)$ coincide, where $A = f_*(\Sigma)$.

The image $f_0(\Sigma_0) = S$ is then a closed immersed real surface whose singularities are ordinary double points (or nodes), and whose tangent bundle is J-invariant: $JTC_0 = TC_0$.

Conversely, using the integrability of almost complex structures on surfaces, such a surface is of the form $S = f_0(\Sigma_0)$, where $[\Sigma_0, f_0]$ is nodal. Moreover, the pair (Σ_0, f_0) is determined by S up to isomorphism, and thus S determines a stable curve $[\Sigma_0, f_0]$ in $\overline{\mathcal{M}}_g(V, J, A)$ where $A = [S]$ and $g = g_a(A)$.

Note that

$$i(A, g_a(A)) = \langle c_1(TV, A)\rangle + g - 1 = \frac{1}{2}(A.A + \langle c_1(TV), A\rangle).$$

We shall denote this number by $d(A)$. Note that

$$\mathrm{ind}\, D^{N_0} = \sum_i \mathrm{ind}\, D^{N_i} = \sum_i i(A_i, g_i) = \langle c_1(TV), A\rangle + g - 1 - m.$$

If f_0 is an embedding, this is also $\frac{1}{2}(A.A + \langle c_1(TV), A\rangle) - m$.

Proposition. *If $\dim(V) = 4$ and $[\Sigma_0, f_0]$ is a J-holomorphic embedding, then every simple J-curve $[\Sigma, f]$ in $\overline{\mathcal{M}}_g(V, J, A)$ is also an embedding. In particular, this applies to every curve sufficiently close to $[\Sigma_0, f_0]$.*

Remark. Presumably, the same result holds in all dimensions, but I do not know how to prove it in the nonintegrable case.

Proof. The adjunction formula above implies that

$$\sum_{s \in \mathrm{Sing}(f(\Sigma))} = \#(\text{nodes of } \Sigma).$$

Thus necessarily each node is embedded, and there is no other singularity on C which means that f is an embedding.

5.3. Automatic regularity

Another aspect of the positivity of intersections is the "automatic regularity" of spaces of J-curves under homotopic conditions [G, 2.1.C$_1$] (cf. also [HLS]). We begin by recalling the linear version.

Proposition. *Let L be a complex line bundle over a smooth Rieman, surface of genus g, equipped with an operator $D : \Gamma(L) \to \Omega^{0,1}(L)$ of the type $\overline{\partial} + a$. Then D is onto provide $c_1(L) > 2g - 2$.*

Now let $f_0 : \Sigma_0 \to (V^4, J)$ be a J-holomorphic map defined on a smooth surface of genus g. When applying the proposition to $L = N_0$, the condition $c_1(L) > 2g - 2$ becomes $\langle c_1(TV), A\rangle > 0$ if f_0 is an immersion, and $c_1(f_0^*TV), \Sigma_i\rangle > |df_0^{-1}(0)|$ if f_0 is nonconstant [IS].

More generally, assume that Σ_0 is nodal, with components $\Sigma_1, \cdots, \Sigma_r$, and f_0 is an embedding with distinct tangents near each node. Then D^{N_0} is isomorphic to the product of the $D^{\widetilde{N}_i}$. Using the formula for $C_1(\widetilde{N}_i)$ in 4.2, we get the

Proposition 1. *Assume that* $\dim(V) = 4$, $\Sigma_0 = \cup_i \Sigma_i$ *is nodal of genus g with m nodes, that f_0 is an embedding near the nodes and has no constant component, and*

$$\langle c_1(f_i^*TV), \Sigma_i\rangle > |df_0^{-1}(0)| \quad \forall i = 1, \cdots, r.$$

Then D^{N_0} is onto.

Remark. The existence of a J-holomorphic curve such that $\langle c_1(TV), C_0\rangle > 0$ and C_0 is not exceptional on a closed almost complex 4-dimensional manifold tamed by a symplectic structure ω is quite restrictive: by a theorem of Li and Liu [LL], this implies that (V, ω) is rational or ruled.

6. Fixing points

The results that we have obtained can be generalized to curves containing a given finite subset F in the smooth part. At the linearized level, it means studying the restriction of D^{N_0} to the subspace of sections of N_0 which vanish on $\widetilde{F}$.

The way to do this is explained in [B], Lemma 4. One replaces N_0 by $N_0 \otimes L(F)$, where $L(F)$ is the bundle associated to the divisor F on Σ_0. Then D^{N_0} induces an operator

$$D_F^{N_0} : L_1^p\Gamma(N_0 \otimes L(F)) \to L^p\Omega^{0,1}(N_0 \otimes L(F)).$$

The regularity (L_1^p, L^p) is essential since $D_F^{N_0}$ contains terms of the type $\frac{\overline{z}}{z}$ in a local coordinate near any point of F. Actually, one does this for the restriction N_i of N_0 to each component of Σ_0, obtaining

$$D_{F_i}^{N_i} : L_1^p\Gamma(N_i \otimes L(F_i)) \to L^p\Omega^{0,1}(N_i \otimes L(F_i)).$$

The Chern class is diminished by $|F_i|$, thus we get

Proposition 2. *Under the hypotheses of Proposition 1, let $F \subset \Sigma_0$ be a finite subset such that*

$$|F_i| + |df_i^{-1}(0)| < \langle c_1(f_i^*TV), \Sigma_i\rangle \quad \forall i = 1, \cdots, r.$$

Then $D_F^{N_0}$ is onto.

Combining this with Theorem 1', we get Corollary 2.

7. J-curves and symplectic surfaces in dimension 4

In this section we assume that (V,ω) is a compact symplectic 4-manifold, so that there is a non-empty and contractible subspace $\mathcal{J}_\omega(V) \subset \mathcal{J}(V)$ of ω-positive almost complex structures.

We study the isotopy problem for symplectic surfaces, in relation with J-holomorphic curves.

Then we restrict to the case $(V,\omega) = (\mathbf{CP}^2, \omega_0)$ and we give explicit sufficient conditions on a J-curve (resp. a curve and a finite subset) to satisfy the hypotheses of Corollary 1 (resp. Corollary 2). and finally we apply Corollary 2 to the case of surfaces of degree 3.

7.1. The isotopy problem for symplectic surfaces

Let $S \subset V$ be a symplectic surface, ie a real embedded surface such that $\omega_{|TS}$ never vanishes. Such a surface is connected and canonically oriented, and has a homology class A.

The relation with J-holomorphic curves is the following fundamental observation of Gromov: S is symplectic if and only if there exists an almost complex structure $J \in \mathcal{J}_\omega(V)$ such that $S = f_0(\Sigma_0)$ where $[\Sigma_0, f_0]$ is an embedded J-holomorphic curve. Note that the genus of S is

$$g = g_a(A) = \frac{A.A - c_1(TV), A}{2} + 1.$$

Let us define

$$\overline{\mathcal{M}}(V,J,A) = \overline{\mathcal{M}}_{g(A)}(V,J,A),$$

which we know is a compact space. It contains as an open subset the connected smooth curves:

$$\mathcal{M}(V,J,A) = \mathcal{M}_{g(A)}(V,J,A).$$

If J is integrable, $\overline{\mathcal{M}}(V,J,A)$ sits over the space $\mathcal{D}(V,A)$ of divisors in the class A (which is an algebraic variety), the projection being a homeomorphism from $\mathcal{M}_{g(A)}(V,J,A)$ to the open subset $\mathcal{D}^s(V,A)$ of connected smooth curves.

The space $\mathcal{D}^s(V,A)$ always has a finite number of connected components (which are the same as path-connected components), thus one may hope that $\mathcal{M}_{g(A)}(V,J,A)$ also has a finite number of path-connected components for any fixed $J \in \mathcal{J}_\omega(V)$. More daringly, one may hope that it is also true for for the space $\mathcal{S}(V,A)$ of connected symplectic surfaces in the class A (which are necessarily of genus $g(A)$). In some cases, as for of $\mathbf{CP}^2$, one may even hope that $\mathcal{S}(V,A)$ is connected:

Question. Let d be a positive integer. Are any two symplectic surfaces of degree d in $\mathbf{CP}^2$ symplectically isotopic ?

Equivalently, is every symplectic surface in $\mathbf{CP}^2$ symplectically isotopic to an algebraic curve (necessarily smooth and of the same degree) ?

In general however, it is not true that this is not true that $\mathcal{S}(V,A)$ has a finite number of path-connected components, as shown recently by R. Fintushel and R. Stern [FS]: if V is simply-connected and $\mathcal{S}(V,A)$ contains a torus T with zero self-intersection which can degenerate to a rational curve with a cusp, then there are infinitely many tori T_n in $\mathcal{S}(V,2A)$ which are pairwise

not differentiably isotopic. In fact, all pairs (V, T_n) are differentiably different. Note that the class $2A$ satisfies $\langle c_1(TV), 2A\rangle = 0$ so that the results of this paper do not apply. This phenomenon is related to the possibility of describing a model of birth of J-holomorphic curves.

The question above has a positive answer for $d = 1$ or 2 by [G]. Note that these are the cases where the genus is zero. Also, for $d = 1$, an embedded sphere is always topologically isotopic to a complex line, and it is an open question whether it is differentiably isotopic. For $d > 1$, there are many surfaces of degree $g(d) = \frac{(d-1)(d-2)}{2}$, which are already topologically knotted, which makes the above question more interesting.

In paragraph 7.3 we shall recall Gromov's proof for $d = 1$ and 2, and explain why it does not directlly extend to $d \geq 3$.

7.2. Hypothetical proof of the isotopy property

In this paragraph we give a positive answer to the above question in $\mathbf{CP}^2$, depending on plausible properties of J-curves.

Let $S \subset \mathbf{CP}^2$ be a symplectic surface of degree d. One can find an almost complex structure $J \in \mathcal{J}_{\omega_0}$ such that S is J-holomorphic.

Let (J_t) be a path in $\mathcal{J}_{\omega_0}$ from the standard structure to J. For $A = d[L]$, denote

$$\overline{\mathcal{M}}(t,d) = \overline{\mathcal{M}}(\mathbf{CP}^2, J_t, d)\ ,\ \mathcal{M}(t,d) = \mathcal{M}(\mathbf{CP}^2, J_t, d),$$

and denote by $\overline{\mathcal{M}}^{\text{sing}}(t,d)$ the subset of singular curves, ie

$$\overline{\mathcal{M}}^{\text{sing}}(t,d) = \overline{\mathcal{M}}(t,d) \setminus \mathcal{M}(t,d).$$

Define also the extended spaces

$$\overline{\mathrm{M}}(d) = \bigcup_{0 \leq t \leq 1} \mathcal{M}(t,d) \times \{t\} = \mathrm{M}(d) \cup \overline{\mathrm{M}}^{\text{sing}}(d),$$

and let $\pi : \overline{\mathrm{M}} \to [0,1]$ denote the second projection.

By Gromov's compactness theorem and the automatic regularity, we have the following properties.

Properties

1) $\overline{\mathrm{M}}(d)$ is compact.

2) $\mathrm{M}(d)$ is a topological manifold of real dimension $d(d+3)+1$, with boundary $\mathrm{M}(d,0) \cup \mathrm{M}(d,1)$, and the restriction of π is a topological submersion.

Note that $\mathrm{M}(d,0)$ is the connected space of all smooth algebraic curves of degree d.

Ideal situation

Assume that $\overline{M}^{\text{sing}}(d)$ *never locally disconnects* $\overline{M}(d)$, ie each point of $\overline{\mathrm{M}}^{\text{sing}}(d)$ admits arbitrarily small neighbourhoods U_i in $\overline{\mathrm{M}}^{\text{sing}}(d)$ such that $U_i \cap \mathrm{M}(d)$ is connected (and thus path-connected). This the case if it has the structure of a subcomplex of real codimension 2, which one can hope would follow from a suitable stratification result.

Under this assumption, the properties 1) and 2) above easily imply the path-connectedness of $\mathrm{M}(d)$ and thus the symplectic isotopy of S to an algebraic curve.

An important generalization of this method arises if we consider curves containing a fixed finite subset $F \subset S$ in the image. We define in a similar manner $\overline{\mathcal{M}}(t,d;F)$, $\mathcal{M}(t,d;F)$, $\overline{\mathcal{M}}^{sing}(t,d;F)$, $\overline{\mathrm{M}}(d;F)$,...

Then Gromov's compactness theorem and the "automatic genericity" imply the

Properties

1') $\overline{\mathrm{M}}(d;F)$ is compact.

2') If $|F| < 3d-1$, $\mathrm{M}(d;F)$ is a topological manifold of real dimension $d(d+3)+1-|F|$, with boundary $\mathrm{M}(d,0;F) \cup \mathrm{M}(d,1;F)$, and the restriction of π is a topological submersion.

Generalized ideal situation

Assume the following:

(*) *For a suitable choice of F and the path (J_t), $\overline{M}^{\mathrm{sing}}(d;F)$ never locally disconnects $\overline{M}(d)$.*

Then again the properties 1') and 2') imply the path-connectedness of $\mathrm{M}(d;F)$ and thus the symplectic isotopy of S to an algebraic curve.

7.3. Curves of degree at most 3

When $d \leq 3$, the above hypothetical proof actually works. For $d=1$ or 2 this is due to Gromov.

Degree 1

The key fact is that here there are no singular curves! Thus properties 1) and 2) imply that $\overline{\mathrm{M}}(1)$ is a compact 4-dimensional manifold(with a natural smooth structure) with boundary $\overline{\mathrm{M}}(1,0) \cup \overline{\mathrm{M}}(1,1)$, and that the projection π is a submersion of index 0. Thus $\overline{\mathrm{M}}(1)$ is diffeomorphic to a product $\overline{\mathrm{M}}(1,0) \times [0,1] \approx \mathbf{CP}^{2*} \times [0,1]$, which gives the desired symplectic isotopy of the symplectic surface to a complex line.

Remark. When one fixes the maximal number $3d-1=2$ points, one thus obtain the famous result of Gromov that through any 2 distinct points there passes a unique J-line.

Degree 2.

Here there are singular curves, which are of two types: a union of two distinct J-lines, and a double J-line. The formal complex dimension of the first is 4, and of the second is 2, since actually it is equivalent to the space of simple lines. Note however that as a subspace of $\overline{\mathcal{M}}(2,J)$ it is of complex dimension 4, being the blow up of the space of double lines in $\mathbf{CP}^5$.

Thus when we fix the maximal number $3d-1=5$ points, the maximal complex formal dimension is -1, thus the real dimension is -2. By a classical argument using Sard-Smale (see [MS] for details, and also [B]), this implies that for a generic choice of F and (J_t), $\overline{\mathcal{M}}^{sing}(d,t)$ is always empty. Thus we conclude exactly as for degree 1.

Degree 3: proof of Theorem 3

Note that now the genus is 1. Here we cannot fix enough to avoid singular curves: indeed the "stratum" corresponding to rational curves with one double point, has formal complex dimension

8, thus we would need to fix $9 = 3d$ points, spoiling property 1'). Considering curves of degree 3 passing through 9 points is similar to considering curves in $\mathbf{CP}^2$ blown up at 9 points: then $c_1(A)$ and $c_1(N)$ vanishes, and one can give a model of birth of J-holomorphic curves. What is more, the examples of Fintushel and Stern show that this can actually lead to an infinite number of isotopy classes.

Thus we fix only 8 points. Using the results of [B], one can prove that all other strata than the one corresponding to rational curves with one double point have formal complex codimension at most 7, and thus will not occur for a generic choice of F and (J_t). The crucial case is that of rational curves with one cusp, which has dimension 7 as a space of curves parameterized by $\mathbf{CP}^1$ (although, as a subspace of the space of stable curves of genus 1, it consists of two strata of dimension 8).

Thus if F and the path (J_t) are chosen generically, , a curve in $\mathcal{M}^{\text{sing}}(3, t; F)$ is rational with a double point, and F is contained in its smooth part.

Then Corollary 2 applies, thus:

i) $\overline{\mathrm{M}}(3; F)$ a topological manifold of dimension 3 is a topological manifold, with boundary $\overline{\mathrm{M}}(3, 0; F) \cup \overline{\mathrm{M}}(3, 1; F)$ and the projection to $[0, 1]$ is a submersion.

ii) $\overline{\mathrm{M}}^{\text{sing}}(3; F)$ is a 1-dimensional locally flat submersion.

Thus the non-disconnecting property (*) is again satisfied, which implies Theorem 3.

Comments We have not been able to extend this method above degree 3: already in degree 4, one cannot avoid curves with a cusp, which as stable curves include a constant component which usually forbids the normal genericity.

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